 Roll No. ....

1(CCE.M)2

Mathematics—I

(15)

Time : Three Hours] [Maximum Marks : 300

INSTRUCTIONS

(i) Answers must be written in English.

(ii) The number of marks carried by each question is indicated at the end of the question.

(iii) The answer to each question or part thereof should begin on a fresh page.

(iv) Your answer should be precise and coherent.

(v) The part/parts of the same question must be answered together and should not be interposed between answers to other questions.

(vi) Candidates should attempt **any five** questions out of the 12 questions.

(vii) If you encounter any typographical error, please read it as it appears in the text-book.

(viii) Candidates are in their own interest advised to go through the General Instructions on the back side of the title page of the Answer Script for strict adherence.

(ix) No continuation sheets shall be provided to any candidate under any circumstances.

(x) Candidates shall put a cross (×) on blank pages of Answer Script.

(xi) No blank page be left in between answer to various questions.

(b) Prove that if $a_{rs} X^r X^s$ is invariant, X^r is an arbitrary contravariant vector and a_{rs} is symmetric in all coordinate systems, then a_{rs} are the components of a covariant tensor of the second order. 20

(c) Prove that at the origin of Riemannian coordinates, the Christoffel symbols of both kinds and the first-order partial derivatives of the metric tensor all vanish. 20

11. (a) Prove that a necessary condition for a system of particles to be in equilibrium is that the vector sum of projections of all external forces applied on it in the fundamental plane vanishes and the sum of the moment N of the applied forces about any line perpendicular to the fundamental plane also vanishes. 20

(b) A light rod of length 2b terminated by heavy particles of weights w, W is placed inside a smooth hemispherical bowl of radius a, which is fixed with its rim horizontal. If the particle of weight w rests just below the rim of the bowl, prove that

$$wa^2 = W(2b^2 - a^2).$$ 20

(c) A rod 4 ft long rests on a rough floor against the smooth edge of a table of height 3 ft. If the rod is on the point of slipping when inclined at an angle of 60º to the horizontal, then find the coefficient of friction. 20

12. (a) Suppose a particle is in simple harmonic motion and x its displacement at time t, a is the amplitude of S.H.M. Then derive the expression :

$$x = a\cos\sqrt{u}\ t$$

where u is any constant. 20

3. (a) Show that the following function is continuous at x = 2, but discontinuous at x = 3.

$$f(x) = 4 - 3x \ ; \ 0 < x < 2$$
$$= 2x - 6 \ ; \ 2 < x \le 3$$
$$= x + 5 \ ; \ 3 < x \le 6.$$ 20

(b) Prove that if u and v are two distinct functions of x and y = uv, then

$$\frac{d^n y}{dx^n} = \binom{n}{0}\frac{d^n u}{dx^n}v + \binom{n}{1}\frac{d^{n-1}}{dx^{n-1}}v\frac{dv}{dx} + \ldots.. + \binom{n}{n}u\frac{d^n v}{dx^n}.$$ 20

(c) Verify Cauchy's mean value theorem for $f(x) = x^2 + 2$ and $g(x) = x^3 - 1$ in [1, 2]. 20

4. (a) Prove that if f is a real valued function of two variables defined on a neighbourhood of the point (a, b) and f is differentiable at (a, b), then

(i) f is continuous at (a, b)

(ii) $f_x(a, b)$ and $f_y(a, b)$ both exist. 20

(b) If $\frac{4}{x} + \frac{9}{y} + \frac{16}{z} = 36$, then, obtain by Lagrange's method, the values of x, y, z which make x + y + z minimum. 20

(c) Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y = x^2$ and $y = x$. 20

5. (a) (i) Can the numbers $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ be the direction cosines of any directed line ? Justify your answer.

(ii) Find the direction cosines of the line which is equally inclined to the axes. 20

(b) Find the equation of the plane passing through the points (0, 2, 4), (3, 1, 1) and (2, 0, –1). 20

(c) Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar. Find also the equation of the plane. 20

6. (a) Find the form of the equation $x^2 + y^2 + z^2 - 12x - 6y + 8z - 3 = 0$ referred to new origin $(-6, 3, -9)$ without changing the direction of axes. 20

(b) Find the equation of the sphere through the four points (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c). 20

(c) Find the nature of the following surfaces given by the equations :

(i) $x^2 + 4y^2 - 9z^2 = 36$

(ii) $4x^2 + 6y^2 - 16x + 12y - 24z - 22 = 0.$ 20

7. (a) Solve the given initial value problem :

$\sin x\, dx + y\, dy = 0;\ y(0) = -2.$ 20

(b) Solve :

(i) $y'=\frac{2+y\,e^{xy}}{2y-x\,e^{xy}}$

(ii) Convert $y'=2xy-x$ into exact diff. equation. 20

(c) (i) Solve

(ii) Find the Wronskian of the set $\{e^x, e^{-x}\}$. 20

8. (a) Prove that the general solution of the n^{th} order differential equation $L(y) = \phi(x)$ is $y = y_n + y_p$ where y_p is any particular solution of the equation and y_n is the general solution of the equation $L(y) = 0$. 20

(b) Find the general solution of the equation :

20

(c) Solve the differential equations :

(i) $y'''-zy''-y'+2y=0$

(ii) . 20

9. (a) (i) Show that the points whose position vectors are $\bar{a},\bar{b}$ and $3\bar{a}-2\bar{b}$ are collinear.

(ii) Show that the vectors $\bar{a}-2\bar{b}+3\bar{c}$, $-2\bar{a}-3\bar{b}-4\bar{c}$, $\bar{a}-3\bar{b}+5\bar{c}$ are coplanar. 20

(b) (i) Find the directional derivative of $f(x, y, z) = x^2 - 2y^2 + 4z^2$ at the point (1, 1, –1) in the direction of 2i + j – k.

(ii) If f=(x + y + 1) i + j + (–x – y)k, show that f-curl f = 0. 20

(c) Evaluate $\iint_S (F\cdot n)ds$ where F = yzi + zxj + xyk, where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant. 20

10. (a) In Euclidean space of three dimensions, write down the equations of transformation between rectangular Cartesian coordinates x, y, z and spherical polar coordinates γ, θ, ϕ. Find the Jacobian of the transformation. Where is it zero or infinite ? 20

(xii) No programmable Calculator is allowed.

(xiii) No stencil (with different markings) is allowed.

1. (a) Define a basis of a vector space. Prove that, in a finite dimensional vector space, any two bases have same number of vectors. 20

(b) Let $T = \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$T(x, y, z) = (2x - 3y + 4z, 5x - y + 2z, 4x + 7y)$.

Find the matrix of T relative to the usual basis of $\mathbb{R}^3$. 20

(c) Find the characteristic polynomial of the matrix :

$$A = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}.$$

Verify Cayley-Hamilton theorem for A. 20

2. (a) Prove that the characteristic roots of a Hermitian matrix are all real. Deduce that a characteristic root of a skew Hermitian matrix is either zero or a pure imaginary number. 20

(b) Find all eigen values and the corresponding eigen vectors of the matrix : $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$. 20

(c) Define a positive definite quadratic form. Consider the quadratic form :

$$q(x, y, z) = x^2 + 2y^2 - 4xz - 4yz + 7z^2.$$

Show that q is positive definite. 20

(b) Find the greatest distance that a stone can be thrown inside a horizontal tunnel of 10 ft height with a velocity of projection of 80 ft/sec. Find also the corresponding time of flight. 20

(c) Find the potential energy of a particle attracted towards a fixed point by a force of magnitude $\frac{k^2}{r^n}$, r be the distance from the fixed point and k, n any constants. 20