Total No.of Printed Pages: 4 **Roll No.** ________________

1(CCE-M)6

STATISTICS-I

[23]

Time Allowed : 3 Hours *Maximum Marks : 300*

INSTRUCTIONS

i) Answers must be written in English.

ii) The number of marks carried by each question is indicated at the end of the question.

iii) The answer to each question or part there of should begin on a fresh page.

iv) Your answer should be precise and coherent.

v) The part/parts of the same question must be answered together and should not be interposed between answers to other questions.

vi) Candidates should attempt ***any Five*** *questions choosing atmost* ***Two*** *from each section*

vii) If you encounter any typographical error, please read it as it appears in the text book.

viii) Candidates are in their own interest advised to go through the General Instructions on the back side of the title page of the Answer Script for strict adherence.

ix) No Continuation sheets shall be provided to any candidate under any circumstances.

x) Candidates shall put a cross (X) on blank pages of Answer Script.

xi) No blank page be left in between answer to various questions.

xii) No programmable Calculator is allowed.

xiii) No stencil (with different markings) is allowed.

xiv) In no circumstances help of scribe will be allowed.

SECTION-I

1. a) Let $f(x,y)=k(x+y);0<x<1,\ 0<y<1,\ 0<x+y<1.$ **(35,25)**

(i) Determine the value of k such that f (x,y) is a probability distribution.

(ii) Find $f_x(x)$ and

(iii) Find the joint and marginal distribution of X + Y and X - Y.

b) An urn contains n red and n blue balls. Two balls are removed from the urn together, at random.

i) What is the Sample Space?

ii) What is the Probability of drawing two balls of different colour?

iii) Find the Probability P_n that the balls are the same colour, and hence evaluate $\lim_{n\to\infty} p_n$.

2. a) Let X and Y have joint density $f(x,y) = \begin{cases} \frac{1}{4a^2}\left[1+xy(x^2-y^2)\right], |x|\le a, |y|\le a, a>0 \\ 0, otherwise \end{cases}$

show that the characteristic functions $\phi X+Y(t)=\phi X(t)\phi Y(t)$, although X and Y are dependent variates. **(40,20)**

b) An urn contains N balls numbered 1 through N, where the first D balls are defective and the remaining N-D are non - defective. A sample of size n balls is drawn from the urn. Let A_k be the event that the sample of n balls contains exactly k defectives. Find the probability $P(A_k)$ when the sample is drawn without replacement.

3. a) Examine whether the WLLN holds for the sequence of independent random variables $\{X_k\}$, having $P(X_K = \pm 2^k) = 1/2$ **(25,15,20)**

b) Stating clearly the reason, arrange the following in increasing order of magnitude: $P(A), P(A\cup B), P(A\cap B), P(A)+P(B)$

c) If m students born in 1991 are attending a lecture, what is the probability that at least two of them share a birth day? What difference would it make if they were born in 1992?

4. a) Consider the geometric distribution in the form $P(x)=2^{-X}, x=1,2,3,....$ Show that Chebyshev's inequality gives $P(|X-2|\le 2)>\frac{1}{2}$, while the exact probability is $\frac{15}{16}$.

b) Let X be a random variable denoting the number of tosses required to get two consecutive heads (E_2) when a fair coin is tossed. Show that the p.g.f. of X is $\frac{S^2}{4}\left\{1-\frac{S}{2}-\frac{S^2}{4}\right\}^{-1}$. Show further that, if a fair coin is tossed indefinitely, the event E2 is certain to occur and that the expected number of trials required is 6. **(20,40)**

SECTION-II

5. a) Show that the most general form of a distribution for which the sample harmonic mean is the ML estimator of θ, has p.d.f. (or p.m.f.)

$$f_\theta(x)=\exp\left[\frac{1}{x}\{\theta A'(\theta)-A(\theta)\}-A'(\theta)+C(x)\right].$$ **(30,30)**

b) If x_i (i=1,2,......,n) is an observed random sample from the distribution having p.d.f. $f_\lambda(x)=\frac{\lambda^{k+1}x^k\exp(-\lambda x)}{\Gamma(k+1)}, x>0.$ Where $\lambda>0$ and k is a known constant.

Obtain the ML estimate for λ. Show that the corresponding estimator is biased but consistent and hence obtain its asymptotic distribution.

6. a) Discuss complete sufficient statistics and suggest the method to find MVUE for $e^{-\lambda}$ in Poisson distribution with parameter λ. **(30,30)**

b) Let $X_1, X_2, \ldots, X_n$ be a sample from Gamma distribution with parameters (r, β). Find the shortest length confidence interval for β at level $(1-\alpha)$, based on a sufficient statistic for β.

7. a) Find the most powerful test for testing $H_0 : f(x) = \begin{cases} 2x, if\, 0 < x < \frac{1}{2} \\ 2(1-x)x, if\, \frac{1}{2} \leq x < 1 \end{cases}$ against the alternative $H_1 : f(x) = 1, if\, 0 < x < 1$, based on a sample of size n = 1. Also find the power of the test. **(25,20,15)**

b) Let $X_1, X_2, \ldots\ldots X_n$ be a random sample from p.d.f. $f(x,\theta) = \begin{cases} \theta x^{\theta-1}, & 0 \leq x \leq 1,\ \theta > 0 \\ 0, & elsewhere \end{cases}$ Obtain a level α UMP test for testing $H_0 : \theta = 2$, against $H_1 : \theta > 2$. Further, obtain power function of the test and sketch the power curve.

c) Describe the sign test for testing the location of a probability distribution. What is the parametric counterpart of this test?

8. a) Consider observations from Poisson distribution with parameter λ. Develop a sequential probability ratio test (SPRT) to test $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda = k\lambda_0, k > 1$. Obtain OC and ASN functions. **(30+30)**

b) Let $X_1, X_2, \ldots\ldots X_n$ be a random sample from p.d.f. $f(x,\theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta,\ -\infty < \theta < \infty \\ 0, & elsewhere \end{cases}$. Find an estimator of θ by the method of moments.

SECTION-III

9. a) If $\underset{\sim}{y}$ is a random vector with mean $\underset{\sim}{\mu}$ and covariance matrix Σ and if A is a symmetric matrix of constants, then

(i) Show that E $(\underset{\sim}{y}' A \underset{\sim}{y})$ = tr $(A\Sigma) + \underset{\sim}{\mu}' A \underset{\sim}{\mu}$. Hence

(ii) Considering $\underset{\sim}{y} = (y_1, y_2, \ldots\ldots, y_n)'$ and a suitable application of the result, show that E $(s^2) = \sigma^2$, where $s^2 = \frac{1}{n'-1}\sum_{i=1}^{n} (y_i - \bar{y})^2$. **(30+30)**

b) Let Y_1, Y_2, Y_3 be uncorrelated observations having common variance σ^2 and

$$E(y_1) = \beta_1 + \beta_2$$
$$E(y_2) = \beta_1 + \beta_3$$
$$E(y_3) = \beta_1 + \beta_2.$$

(i) Show that $\lambda_1\beta_1 + \lambda_2\beta_2 + \lambda_3\beta_3$ is estimable if and only if $\lambda_1 = \lambda_2 + \lambda_3$.

(ii) Obtain an unbiased estimate of σ^2.

10. a) When is a two - way ANOVA model said to be mixed ? How will you formulate the different hypotheses of this model ? Describe how the hypotheses are tested. **(25,35)**

b) Let the random vector $\underset{\sim}{v} \sim N_4(\underset{\sim}{\mu}, \Sigma)$, where

$$\underset{\sim}{\mu} = \begin{bmatrix} 2 \\ 5 \\ -2 \\ 1 \end{bmatrix} \Sigma = \begin{bmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{bmatrix}$$

If $\underset{\sim}{v}$ is partitioned as $\underset{\sim}{v} = (y_1, y_2, x_1, x_2)'$, then $\underset{\sim}{\mu}_y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \underset{\sim}{\mu}_x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$,

$\Sigma_{yy} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \Sigma_{yx} = \begin{bmatrix} 3 & 3 \\ -1 & 2 \end{bmatrix}$ and $\Sigma_{xx} = \begin{bmatrix} 6 & -3 \\ -3 & 7 \end{bmatrix}$. Then obtain E $(\underset{\sim}{y} / \underset{\sim}{x})$ and Cov $(\underset{\sim}{y} / \underset{\sim}{x})$.

11. a) If $\underset{\sim}{X} \sim Np(\underset{\sim}{\mu}, \Sigma)$ (MVND), then show that its moment generating function is given by $M_X(t) = e^{t'\mu + \frac{1}{2}t'\Sigma t}$. **(30,30)**

b) Define Hotelling's T^2. Show that it is invariant under non - singular linear transformations. Mention some uses of the T^2 statistic.

12. a) Explain the concept of partial correlation. Show in the usual notations that

$r_{12.3} = \dfrac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r^2_{13})(1-r^2_{23})}}$ Further, if r_{12} and r_{13} are given, then show that r_{23} must lie in the range: **(30,30)**

$$r_{12}r_{13} \pm \left(1 - r_{12}^2 - r_{13}^2 + r_{12}^2 r_{13}^2\right)^{1/2}.$$

b) Let $\underset{\sim}{X} \sim Np(\underset{\sim}{\mu}, \Sigma)$ and let $\underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{X}^{(1)} \\ \underset{\sim}{X}^{(2)} \end{bmatrix} \underset{\sim}{\mu} = \begin{bmatrix} \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{\mu}^{(2)} \end{bmatrix} \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

Where $\underset{\sim}{X}^{(1)}_{q\times 1}$. Show that $\underset{\sim}{X}^{(1)} \sim Nq(\underset{\sim}{\mu}^{(1)}, \Sigma_{11})$, even if $\underset{\sim}{X}^{(1)}$ and $\underset{\sim}{X}^{(2)}$ are not independent.