Total No. of Printed Pages-6 **Roll No.** ____________

1(CCE-M)6

MATHEMATICS - II

[15]

Time Allowed -3 Hours *Maximum Marks-300*

INSTRUCTIONS

i) *Answers must be written in English*

ii) *The number of marks carried by each question is indicated at the end of the question.*

iii) *The answer to each question or part thereof should begin on a fresh page.*

iv) *Your answer should be precise and coherent.*

v) *The part/parts of the same question must be answered together and should not be interposed between answers to other questions.*

vi) *Candidates should attempt any **five** questions*

vii) *If you encounter any typographical error, please read it as it appears in the text book.*

viii) *Candidates are in their own interest advised to go through the General Instructions on the back side of the title page of the Answer Script for strict adherence.*

ix) *No Continuation sheets shall be provided to any candidate under any circumstances.*

x) *Candidates shall put a cross (X) on blank pages of answer Script.*

xi) *No blank page be left in between answer to various questions.*

xii) *No programmable Calculator is allowed.*

xiii) *No stencil (with different markings) is allowed.*

xiv) *In no circumstances help of scribe will be allowed.*

Section - A

1. a) Suppose G is a group and $g \neq e$ is an element of G. Under what conditions on g is there a homomorphism $f : \mathbb{Z}_{15} \to G$ *with* $f([1]) = g$? Under what conditions on G is there an injective homomorphism $f : \mathbb{Z}_{15} \to G$? **(20)**

b) Show that every group of order 85 is cyclic **(20)**

c) Using Sylow Theorems, show that in a group of order *pq* with two primes *p,q* and $p < q$, there is only one subgroup of order q. Further show that this subgroup is normal. **(20)**

2. Let G be an abelian group of order n.

a) If $f: G \rightarrow \mathbb{C}$ is a function, then prove that for all $h \in G$,

$$\sum_{g\in G} f(g) = \sum_{g\in G} f(hg)$$ **(20)**

b) Let $\mathbb{C}^*$ be the multiplicative group of non - zero complex numbers and suppose $f: G \rightarrow \mathbb{C}^*$ is a homomorphism. Prove that $\sum_{g\in G} f(g) = 0$ *or* $\sum_{g\in G} f(g) = n$ **(30)**

c) If $f: G \rightarrow \mathbb{C}^*$ is any homomorphism, then prove that $\sum_{g\in G} |f(g)| = n$ **(10)**

3. Let $R_n = \mathbb{Z}_n[x]$ be the polynomial ring over integers modulo n.

a) In the field $R_2/(x^2+x+1)$ let α be the image of x and compute (in reduced form) $(1+\alpha-\alpha^2)^{-1}$ **(30)**

b) In the field $R_3/(x^3-x+1)$, let α be the image of x and compute (in reduced form) $(1+\alpha)(2+\alpha-\alpha^2)$ **(30)**

4. Let (X, d) be a metric space

a) Prove that $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ is metric on X. **(10)**

b) Let A be a nonempty subset of X. For any $x \in A$, we define $d(x,A) = \inf\{d(x,y) : y \in A\}$.

Prove that for any $x, y \in X$, *we have* $|d(x,A) - d(y,A)| \le d(x,y)$ **(30)**

c) Let $a, b \in R, a < b$ *and let*

$$C([a,b]) = \{f : [a,b] \rightarrow \mathbb{R} : f \text{ is continuous on } [a,b]\}$$

Show that the map $d : C([a,b]) \times C([a,b]) \rightarrow \mathbb{R}_+$ defined by

$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$ is a metric on

$C([a,b])$. *Therefore,* $(C([a,b]), d)$ is a metric space **(20)**

5. a) Prove that if f is holomorphic in the annulus

$$U = \{z \in \square : r < |z-a| < R\},$$

then the integral $I = \int_{|z-a|<\rho} f(z)\,dz$

has the same value for all ρ with $r < \rho < R$. **(30)**

b) Compute the value of the integral

$$I = \int_{\gamma} \frac{z+1}{z(z-1)}\,dz$$

Where γ is the circle $|z-2| = \sqrt{2}$, traversed counterclockwise. **(30)**

6. a) Find the radius of convergence for the series

$$\sum_{n\geq 1} \frac{n^n z^n}{n!} \text{ and } \sum_{n\geq 1} 2^{-n^2} z^n$$ **(30)**

b) For any positive integer n, prove that

$$\int_0^{2\pi} (1+2\cos t)^n (\cos nt)\,dt = 2\pi$$ **(30)**

7. a) Let $f_n(x) = x^n$ for $x \in [0,1]$. Then $\{f_n\}$ converge pointwise but not uniformly.

on [0,1]. Let $g : [0,1] \to \square$ be a continuous function such that $g(1) = 0$. Prove that $\{g(x)x^n\}$ converge uniformly on [0,1] **(30)**

b) For $f(t) = 2t^2 + 1$, $\alpha(t) = t + [3t] on [0,1]$ and P is the partition of [0,1] consisting of 4 subintervals of equal length, find $U(P, f, \alpha)$ and $L(P, f, \alpha)$. Further.

calculate $\int_0^1 f\,d\alpha$ if it exists. **(30)**

8. a) Use separation of variables to solve the wave equation $u_{xx} - C^2 u_{tt} = 0$ with initial conditions $u(x,0) = f(x), u_x(x,0) = g(x)$. **(20)**

b) Use the method of characteristics to solve the wave equation and verify that these solutions agree with those obtained in part (a). **(20)**

c) Find the general integral of the PDE :

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = 3.$$ **(20)**

SECTION - B

1. a) Derive the Euler-Lagrange equations for the motion of a rigid body under a conservative force. **(20)**

b) Consider a simple pendulum of length l vertically suspended from a point. If a horizontal force of F $\cos(\omega t)$ is applied at that point, prove that the pendulum settles into a periodic motion. Find the period of this motion. **(20)**

c) Consider a circular turn table inclined at angle of θ to the horizontal, which has a constant angular velocity ω about the fixed centre. A block of wood of mass m lies at rest relative to the turn table. What is the smallest value of θ for which the block will begin to slide at atleast one point in the period of rotation. Assume that the coefficient of friction between the table and the block is μ. **(20)**

2. a) Describe the test for imcompressibility of a flow. How are the stream lines derived from a given flow. **(20)**

b) Derive Navier Stokes equation for an incompressible flow. **(20)**

c) Give the definitions of the terms source and sink. How do the flow equations change in the presence of these phenomena. **(20)**

3. a) Find a real root of the polynomial $p(x) = x^5 + 3x^3 - 1$, correct to $\in = 0.05$ using the regula falsi method as well as the Newton-Raphson method. Which has higher convergence and why? **(30)**

b) Derive the cubic spline interpolation for the function given by the following table

x	1	3	5	7
$f(x)$	2	6	24	120

Use this to find the values of $f(2)$ and $f(6)$. **(30)**

4. a) Solve the I.V.P. $y=(l+y)^2$, $y(0)=1$, in the interval $0\leq x\leq 1$ by using the Runge Kutta method of order 2 by using h = 0.2 **(30)**

b) State the n-point Gaussian quadrature formula for weight $w(x)=1$ in the interval [-1,1]. Derive the error of approximation for any continuous function $f\in C^{n+1}([-1,1])$. **(30)**

5. a) A token collector collects tokens number 1,2,.....n all of which are equally likely. What is the expected number of tokens collected before token number i is collected.What are the expected number of tokens before the collection is complete (viz., at lease one token of every number is collected). **(30)**

b) Consider a random walk in the set of integers, where the starting point, is 0 and the steps taken at any point are independently chosen from either of 0, $\pm 1, \pm 2$ with equal probability. Let X be the random variable that holds the value of the current position after n steps. Find E[X] and V ar [X]. Hence, use the central limit theorem to find the limiting distribution of X as $n\to\infty$. **(30)**

6. a) Which continuous distribution has the memory-less property? Give a proof of this statement. **(10)**

b) What is the difference between Multiple correlation coefficient and Rank correlation coefficient? Give an example where these values might be different. **(20)**

c) Describe the least squares approach to linear regression. Derive the optimal parameters from first principles. **(30)**

7. a) Let X,Y be random variables having the following joint distribution

$$f(x,y)=\begin{cases}(2\pi)^{-1} & x^2+y^2\le 1\\ (2\pi)^{-1} & (x-1)^2+y^2\le 1\\ 0 & otherwise\end{cases}$$

Find the marginal distributions of X and Y. Calculate the value of P[X+Y $\geq$ 1]. Calculate the condition probability density function $f_{Y/X}(y)$. **(20)**

b) Out of a batch of 30 LED lamps, 11 were found to be defective. Test the hypothesis that the probability of a lamp being defective is 1/3 at the both levels of significance α=0.05 and α=0.01. (Assume that, for n=1 degree of freedom, $\xi^2_{0.95}$=3.84 and $\xi^2_{0.99}$=6.63) **(20)**

c) i) Define the power of a test. How does it differ from the level of significance of a test? **(10)**

ii) Let the population variance be σ_p^2 and let a sample be drawn from this population with sample variance be σ_s^2. Design a test for the equality of σ_s and σ_p. Would your answer change if we only have the variances of two different samples. **(10)**

8. a) Derive the Kuhn-Tucker conditions for the solution of a non-linear progra ming problem. **(20)**

b) The arrival of customers at a hair dressing salon follows a Poisson process at the rate of 4 per hour. The salon has two hair dressers, each of whom complete their services in an exponential distribution with the rate $\lambda=0.1$ per minute. Answer the following questions.

i) What is the average waiting time of a customer ? **(10)**

ii) How much free time does a hair dresser get on an average ? **(10)**

iii) Describe the steady-state solution of this queue. If the number of hair dressers increases by one, how would the steady state solution change ? **(20)**
